Weight Ideals Associated to Regular and Log-Linear Arrays

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Abstract

Certain weight-based orders on the free associative algebra $R = k\langle x_1, \ldots, x_t\rangle$ can be specified by $t \times \infty$ arrays whose entries come from the subring of nonnegative elements in a totally ordered field. Such an array A satisfying certain additional conditions produces a partial order on R which is an admissible order on the quotient R/I_A, where I_A is a homogeneous binomial ideal called the *weight ideal* associated to the array and whose structure is determined entirely by A. This article discusses the structure of the weight ideals associated to two distinct sets of arrays whose elements define admissible orders on the associated quotient algebra.

Key words: Noncommutative Gröbner Bases, Gröbner Bases, Admissible Orders

1 Introduction

Work over the past two decades has extended the theory of Gröbner bases to various noncommutative algebras (Green, 2000; Madlener & Reinert, 1997; Nordbeck, 2001; Mora, 1994). Before a Gröbner basis for an ideal of a k-algebra $\mathcal{A}$ can be constructed, where k is a field, an admissible order on a multiplicative basis of $\mathcal{A}$ is required. Following Green (1996), we say that $\mathcal{A}$ has a *Gröbner basis theory* when an admissible order exists on a multiplicative basis of $\mathcal{A}$. In Hinson (2010), E. Hinson adapted the theory of position-dependent weighted orders to define a length-dominant partial order on the set of words in the free associative algebra $R = k\langle x_1, \ldots, x_t\rangle$, including the trivial word, which produces an admissible order on a quotient of R. In this construction, the partial order on R is specified by a $t \times \infty$ array A whose entries come from the subring consisting of the positive elements of a totally ordered field, and the quotient

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is by a homogenous binomial ideal I_A whose elements are determined by the partial order given by A. This gives rise to two immediate questions. First, given an array A that defines an admissible order on a quotient R/I_A, what is the algebra that is determined, or more specifically, what is the structure of the ideal I_A? Second, given two arrays A and B which define orders $\succ_A$ and $\succ_B$ on R/I_A and R/I_B respectively, even when $R/I_A = R/I_B$ it is not necessarily the case that $\succ_A = \succ_B$. Under what circumstances does $\succ_A = \succ_B$? This paper describes results concerning the first of these two questions for two distinct families of admissible arrays. In this introductory section, we review the relevant definitions and results from Hinson (2010) and we make the preceding general statements precise. Our primary objects of interest are defined in Definitions 5 and 6. The results on which the remainder of the paper relies are given in Theorems 7 and 8. In what follows, let $R = k\langle x_1, \ldots, x_t\rangle$ denote the free associative algebra, let $S_{>0}$ denote the positive elements of a totally ordered field, and let $\mathcal{M}_{t\times\infty}(S_{>0})$ denote the set of $t \times \infty$ arrays with entries in $S_{>0}$. The following two definitions are adopted from Green (1996).

Definition 1 *Let $\mathcal{B}$ be a k–basis of an algebra $\mathcal{A}$. $\mathcal{B}$ is a* multiplicative basis *for $\mathcal{A}$ if*

$$b, b' \in \mathcal{B} \Rightarrow b \cdot b' \in \mathcal{B} \text{ or } b \cdot b' = 0.$$

We will have occasion to refer to the nontrivial elements of $\mathcal{B}$, which we denote by $\mathcal{B}^\times$.

Definition 2 *A total order $\succ$ on a multiplicative basis $\mathcal{B}$ of $\mathcal{A}$ is an* admissible order *on $\mathcal{B}$ if*

- *$\succ$ is a well-order on $\mathcal{B}$,*
- *for all b_1, b_2, $b_3 \in \mathcal{B}$ such that $b_1b_3 \neq 0$ and $b_2b_3 \neq 0$, if $b_1 \succ b_2$, then $b_1b_3 \succ b_2b_3$,*
- *for all b_1, b_2, $b_3 \in \mathcal{B}$ such that $b_3b_1 \neq 0$ and $b_3b_2 \neq 0$, if $b_1 \succ b_2$, then $b_3b_1 \succ b_3b_2$, and*
- *for all b_1, b_2, b_3, $b_4 \in \mathcal{B}$, if $b_1 = b_2b_3b_4$, then $b_1 \succeq b_3$.*

Commonly used admissible orders for Gröbner basis calculations on noncommutative algebras are the left length-lexicographic order or the right length-lexicographic order (Green, 1996). We specify a position-dependent weighted order on words in the free algebra using a $t \times \infty$ array to define a weight function as described in the following definition.

Definition 3 *Let $A = (a_{i,j}) \in \mathcal{M}_{t\times\infty}(S_{>0})$. A gives a monomial weighting $\sigma_A : \mathcal{B}^\times \to S_{>0}$ by*

$$\sigma_A(x_{u_0}x_{u_1}\cdots x_{u_{l-1}}) = \prod_{j=0}^{l-1} a_{u_j,j}$$

for a given monomial $x_{u_0}x_{u_1}\cdots x_{u_{l-1}} \in R$. *The function* σ_A *is the* weight function *associated to* A.

Note that for computational convenience we index the columns of an array starting with 0 rather than 1. When the array A is clear, we will suppress it from the notation and write the associated weight function σ_A simply as σ. For the remainder of this section, fix an array $A \in \mathcal{M}_{t\times\infty}(S_{>0})$ and associated weight function σ. In order to discuss the weight of the product of two words, we identify a translated version of the weight function associated to A by

$$\sigma_k(x_{u_0}x_{u_1}\cdots x_{u_{l-1}}) = \prod_{j=0}^{l-1} a_{u_j,j+k},$$

where $k \in \mathbb{N}$. We consider $\sigma(\omega) = \sigma_A(\omega) = \sigma_{A,0}(\omega)$. Let $|\omega|$ denote the length of ω. Given ω and λ such that $|\omega| = k$,

$$\sigma(\omega\lambda) = \sigma(\omega)\cdot\sigma_k(\lambda).$$

This gives rise to the following equivalence relation.

Definition 4 *Define the relation* $\succ_\sigma$ *on* $\mathcal{B}^\times$ *by*

$$\omega_1 \succ_\sigma \omega_2 \iff |\omega_1| > |\omega_2|, \text{ or } |\omega_1| = |\omega_2| \text{ and } \sigma(\omega_1) > \sigma(\omega_2).$$

Let Γ denote the set of pure homogeneous binomial differences $\omega_1 - \omega_2$, where $\omega_1, \omega_2 \in \mathcal{B}^\times$, $|\omega_1| = |\omega_2|$, and $\sigma(\omega_1) = \sigma(\omega_2)$.

Definition 5 *The ideal* $I_A = \langle\Gamma\rangle$ *is the* weight ideal *associated to* A.

Definition 6 A *is an* admissible *array if for every pair* $\omega_1, \omega_2 \in \mathcal{B}^\times$ *with* $|\omega_1| = |\omega_2|$,

(1) for all $k \geq 0$, *if* $\sigma_k(\omega_1) > \sigma_k(\omega_2)$, *then* $\sigma_{k+1}(\omega_1) > \sigma_{k+1}(\omega_2)$, *and*
(2) for all $k \geq 0$, *if* $\sigma_k(\omega_1) = \sigma_k(\omega_2)$, *then* $\sigma_{k+1}(\omega_1) = \sigma_{k+1}(\omega_2)$.

The following theorem illustrates that the second part of Definition 6 is in fact unnecessary.

Theorem 7 *Let* $A \in \mathcal{M}_{t\times\infty}(S_{>0})$ *be an array with associated weight function* σ. *The following are equivalent:*

(1) A *is an admissible array;*
(2) for all $k \geq 0$ *and for all* $\omega_1, \omega_2 \in \mathcal{B}^\times$ *such that* $|\omega_1| = |\omega_2|$, $\sigma_k(\omega_1) > \sigma_k(\omega_2)$ *if and only if* $\sigma_{k+1}(\omega_1) > \sigma_{k+1}(\omega_2)$.

Admissible arrays define an admissible order on the quotient R/I_A.

Theorem 8 *An array $A \in \mathcal{M}_{t\times\infty}(S_{>0})$ with associated weight function σ is an admissible array if and only if $\succ_\sigma$ is an admissible order on $\mathcal{B}_\sigma \subseteq R/I_A$, where $\mathcal{B}_\sigma$ is the image of $\mathcal{B}$ in R/I_A under the projection $R \to R/I_A$.*

Definition 9 *A is said to be* degenerate *if there exists i, j, $1 \leq i \neq j \leq t$, such that $\sigma(x_i) = \sigma(x_j)$.*

We will assume in what follows that all arrays considered are nondegenerate, for if $\sigma(x_i) = \sigma(x_j)$ for some $i, j \in \{1, \ldots, t\}$ where $i \neq j$, then $x_i - x_j \in I_A$ and $k\langle x_1, \ldots, x_t\rangle / I_A \simeq k\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_t\rangle / \langle I'_A\rangle$ where A' is the array obtained from A by deleting the i^{th} row.

2 Weight Ideals Associated to Regular Arrays

In Hinson (2010), E. Hinson described two sets of admissible arrays. We begin by studying the first of these, the set of regular arrays.

Definition 10 *An array A is* regular *if A has rank 1.*

The set of linear arrays is a subset of the set of regular arrays which will be used later on to construct the set of log-linear arrays.

Definition 11 *An array A is* linear *if for all $i \geq 1$, $A_{(i)} = d \cdot A_{(i-1)}$ for some fixed $d \in S_{>0}$. The fixed scalar d is referred to as the* slope *of the array.*

Example 12 *The array*

$$A = \begin{pmatrix} 2 & 6 & 18 & \cdots \\ 3 & 9 & 27 & \cdots \\ 4 & 12 & 36 & \cdots \end{pmatrix}$$

is a linear array with slope $d = 3$.

The weight ideal associated to a regular array contains the commutator ideal $\mathcal{C} = \langle x_i x_j - x_j x_i | 1 \leq i \neq j \leq t\rangle$, and thus is never trivial (Hinson, 2010).

Definition 13 *The* support *of a word ω is the set*

$$supp(\omega) = \{x_i | i \in \{1, \ldots, t\} \text{ and } x_i \text{ occurs in } \omega\}.$$

Definition 14 *The* frequency of x in ω *is the number of times that x occurs in ω and is written $\#(x, \omega)$.*

Definition 15 *Let $f \in R$ and $G = \{g_1, g_2, \dots\} \subset R$. We say that f is an* algebraic consequence of G *if $f = \sum_{g \in G} c_i u_i g_i v_i$, where $c_i \in k$, u_i, $v_i \in R$, and only finitely many $c_i \neq 0$.*

Suppose A is a regular array with first column $[a_{1,0}, \dots, a_{t,0}]^T$, where $a_{i,0} \in \mathbb{N}$ and at least one of $(a_{i,0}, a_{j,0}) \neq 1$, where $(a_{i,0}, a_{j,0})$ denotes the greatest common divisor of $a_{i,0}$ and $a_{j,0}$ and $1 \leq i \neq j \leq t$. Let $\omega_1 = x_{u_0} \cdots x_{u_{l-1}}$ and $\omega_2 = x_{v_0} \dots x_{v_{l-1}} \in \mathcal{B}$ such that $\omega_1 - \omega_2 \in I_A$. Then we have

$$\prod_{i=0}^{l-1} a_{u_i i} = \prod_{i=0}^{l-1} a_{v_i i},$$

and each $a_{u_i i}$ and $a_{v_i i}$ can be written as scalar multiples of $a_{u_i 0}$ and $a_{v_i 0}$ respectively:

$$\prod_{i=0}^{l-1} d_i a_{u_i 0} = \prod_{i=0}^{l-1} d_i a_{v_i 0}.$$

Factoring out and canceling the common d_i's reduces the equation to

$$\prod_{i=0}^{l-1} a_{u_i 0} = \prod_{i=0}^{l-1} a_{v_i 0}. \tag{1}$$

Equation (1) does not depend on the how the variables were ordered in ω_1 and ω_2; in particular, by factoring out and canceling any terms $a_{u_i 0} = a_{v_j 0}$ common to both sides of the equation, one obtains the reduced expression

$$\prod_{k=0}^{n} a_{u_k 0} = \prod_{k=0}^{n} a_{v_k 0}. \tag{2}$$

In this expression, $a_{u_m 0} \neq a_{v_{m'} 0}$ for all u_m and $v_{m'}$. Note that we have not cancelled any common divisors of the $a_{u_i,0}$, we have only cancelled those $a_{u_i,0}$'s and $a_{v_j,0}$'s for which $a_{u_i,0} = a_{v_j,0}$. Since each $a_{u_m 0}$ and $a_{v_{m'} 0}$ corresponds to the weight assigned to an individual letter in $\{x_1, \dots, x_t\}$, this equation describes a homogeneous binomial difference $\omega_1' - \omega_2' \in I_A$ in which no letter that occurs in ω_1' will occur in ω_2'.

Definition 16 *A homogeneous binomial difference $\omega_1 - \omega_2 \in I_A$ for which*

$$supp(\omega_1) \bigcap supp(\omega_2) = \emptyset$$

will be referred to as a homogeneous binomial difference of disjoint support*.*

Homogeneous binomial differences of disjoint support may arise as algebraic consequences of other homogeneous binomial differences of disjoint support.

For example, suppose

$$A = \begin{pmatrix} 2 & 4 & 8 & \cdots \\ 3 & 6 & 12 & \cdots \\ 4 & 8 & 16 & \cdots \\ 6 & 12 & 24 & \cdots \end{pmatrix}.$$

This array A is linear with slope 2. Consider the homogeneous binomial difference $x_3x_2x_3x_2 - x_4x_1x_4x_1$. Since

$$\sigma(x_3x_2x_3x_2) = \sigma(x_4x_1x_4x_1) = 9216,$$

we must have $x_3x_2x_3x_2 - x_4x_1x_4x_1 \in I_A$. Neither word in this homogeneous difference shares a letter with the other, so $x_3x_2x_3x_2 - x_4x_1x_4x_1$ is a homogeneous binomial difference of disjoint support. Furthermore,

$$x_3x_2x_3x_2 - x_4x_1x_4x_1 = (x_3x_2 - x_4x_1)x_3x_2 + x_4x_1(x_3x_2 - x_4x_1),$$

so $x_3x_2x_3x_2 - x_4x_1x_4x_1$ is a homogeneous binomial difference of disjoint support which arises as an algebraic consequence of a homogeneous binomial difference of disjoint support consisting of words of lesser length.

Definition 17 *Let $\omega_1 - \omega_2$ be a homogeneous binomial difference of disjoint support. $\omega_1 - \omega_2$ is* minimal *if any expression*

$$\omega_1 - \omega_2 = \sum_{i=1}^{n} \alpha_i(u_i - v_i)\beta_i$$

for $\omega_1 - \omega_2$ as a sum of homogeneous binomial differences has at least one difference $u_i - v_i$ such that $|u_i| = |v_i| = |w_1|$. $\mathcal{M}_A$ will be used to denote the set of minimal length homogeneous binomial differences of disjoint support associated to A.

In other words, a minimal homogeneous binomial difference of disjoint support is one which cannot be realized as an algebraic consequence of homogeneous binomial differences of disjoint support consisting of words of lesser length.

Any element of I_A may be decomposed over the set of commutators $\{x_ix_j - x_jx_i | 1 \leq i \neq j \leq t\}$ and the set of homogeneous binomial differences of disjoint support.

Lemma 18 *Let $\omega_1 - \omega_2$ be a homogeneous binomial difference in I_A. Then $\omega_1 - \omega_2 = \sum_{i=1}^{n} \alpha_i(u_i - v_i)\beta_i$, where each homogeneous binomial difference $u_i - v_i$, $1 \leq i < n$ is a commutator and $u_n - v_n$ is a homogeneous binomial difference of disjoint support.*

PROOF. Suppose $\omega_1 - \omega_2 \in I_A$. We proceed by induction. The base case when $l = 2$ is established trivially. Assume now that the induction hypothesis holds for homogeneous binomial differences consisting of words of length $l-1$ and suppose $|\omega_1| = |\omega_2| = l$. Write $\omega_1 = x_{u_0} \dots x_{u_{l-1}}$ and $\omega_2 = x_{v_0} \dots x_{v_{l-1}}$. Let $i \in \{0, \dots, l-1\}$ be the least value for which $x_{u_0} = x_{v_i}$ (if no such value exists, we are done). By inserting the expression

$$-x_{v_0} \dots x_{v_{i-2}} x_{v_i} x_{v_{i-1}} x_{v_{i+1}} \dots x_{v_{l-1}} + x_{v_0} \dots x_{v_{i-2}} x_{v_i} x_{v_{i-1}} x_{v_{i+1}} \dots x_{v_{l-1}},$$

we obtain

$$\omega_1 - x_{v_0} \dots x_{v_{i-2}} x_{v_i} x_{v_{i-1}} x_{v_{i+1}} \dots x_{v_{l-1}} + x_{v_0} \dots x_{v_{i-2}} x_{v_i} x_{v_{i-1}} x_{v_{i+1}} \dots x_{v_{l-1}} - \omega_2,$$

which is equal to

$$\begin{multline} \omega_1 - x_{v_0} \dots x_{v_{i-2}} x_{v_i} x_{v_{i-1}} x_{v_{i+1}} \dots x_{v_{l-1}} + \\ x_{v_0} \dots x_{v_{i-2}} \left(x_{v_i} x_{v_{i-1}} - x_{v_{i-1}} x_{v_i} \right) x_{v_{i+1}} \dots x_{v_{l-1}}. \quad (3) \end{multline}$$

In the second term in expression (3), x_{v_i} occurs in the $i-1^{st}$ position. The third and fourth terms in Equation 3 have been expressed as (left and right) multiples of the commutator $x_{v_i} x_{v_{i-1}} - x_{v_{i-1}} x_{v_i}$. Iterating this process i times results in the expression

$$\omega_1 - x_{v_i} x_{v_0} \dots x_{v_{i-1}} x_{v_{i+1}} \dots x_{v_{l-1}} + \sum_{k=1}^{i-1} \alpha_k (x_{v_i} x_{v_{i-k}} - x_{v_{i-k}} x_{v_i}) \beta_k, \quad (4)$$

where $\alpha_k = x_{v_0} \dots x_{v_{i-k-1}}$ and $\beta_k = x_{v_{i-k+1}} \dots x_{v_{l-1}}$.

Since $x_{u_0} = x_{v_i}$, the difference of the first two terms in 4 can be rewritten as

$$x_{u_0} \left(x_{u_1} \dots x_{u_{l-1}} - x_{v_1} \dots x_{v_{l-1}} \right).$$

The expression in parentheses consists of monomials of length $l-1$ which is an algebraic consequence of the commutators and a homogeneous binomial difference of disjoint support. Rearranging and renaming terms as needed gives the desired result. $\square$

Theorem 19 *Let A be a regular array. The weight ideal I_A associated to a regular array A is generated by the union of the set of commutators $\{x_i x_j - x_j x_i | 1 \leq i \neq j \leq t\}$ and $\mathcal{M}_A$.*

PROOF. Fix a homogeneous binomial difference $\omega_1 - \omega_2 \in I_A$. By iterating the algorithm described in the proof of Lemma 18, $\omega_1 - \omega_2 \in I_A$ can be reduced to an algebraic consequence of the commutators plus a single, perhaps trivial, homogeneous binomial difference of disjoint support $\omega_1' - \omega_2'$. To see this, note that each iteration of the algorithm produces in the sum a difference of commutators and a homogeneous binomial difference of shorter length than in the previous iteration in which a letter common to each word has been extracted. We may continue the algorithm until either the next iteration is over a commutator or there are no common letters to extract. In the first case, we are done, and in the second case, if $\omega_1' - \omega_2'$ is minimal, we are also done. If $\omega_1' - \omega_2'$ is not a minimal homogeneous binomial difference, then by definition it is an algebraic consequence of minimal homogeneous binomial differences of disjoint support. $\square$

Having obtained a description of the generators of I_A, we will next show that when A is regular, I_A is finitely generated. We include the following lemma to describe the means by which a disjoint homogeneous binomial difference which contains another difference as scattered subwords can be decomposed over that subdifference.

Lemma 20 *Let $\omega_1 - \omega_2 \in \mathcal{M}_A$ and suppose $\lambda_1 - \lambda_2$ is a homogeneous binomial difference of disjoint support such that ω_1 occurs as a scattered subword in λ_1 and ω_2 occurs as a scattered subword in λ_2. Then*

$$\lambda_1 - \lambda_2 = (\omega_1 - \omega_2)\alpha + \omega_2(\alpha - \beta) + \sum_{i=1}^{n} \alpha_i(\gamma_i - \zeta_i)\beta_i,$$

where $\alpha - \beta$ is a homogeneous binomial difference of disjoint support and $\gamma_i - \zeta_i$ is a commutator for each i, $1 \leq i \leq n$.

PROOF. The algorithm of Lemma 18 may be modified to move any letter that occurs in a word in a homogeneous binomial difference in I_A either forward or backwards to the desired position, resulting in a decomposition

$$\lambda_1 - \lambda_2 = \omega_1\alpha - \omega_2\beta + \sum_{i=1}^{n} \alpha_i(\gamma_i - \zeta_i)\beta_i,$$

where $\gamma_i - \zeta_i$ is a commutator, $1 \leq i \leq n$. The result then follows.

Theorem 21 *Let A be a regular array. The associated weight ideal I_A is finitely generated.*

PROOF. By Theorem 19, I_A is generated by the union of the set of commutators and the set $\mathcal{M}_A$ of minimal homogeneous binomial differences of disjoint

support. The set of commutators is clearly finite. It remains to demonstrate that $\mathcal{M}_A$ is also finite. Assume the contrary. Then there exists some partition of $X = \{x_1, \ldots, x_t\}$ into two sets X_1, X_2 such that there are infinitely many minimal disjoint homogeneous binomial differences $\omega_1 - \omega_2$ in which $\text{supp}(\omega_1) \subseteq X_1$ and $\text{supp}(\omega_2) \subseteq X_2$. Let $\mathcal{D} = \{\omega_1 - \omega_2 \in \mathcal{M}_A | \text{supp}(\omega_1) \in X_1, \text{supp}(\omega_2) \in X_2\}$ and let $\omega_1 - \omega_2 \in \mathcal{D}$ such that $|\omega_1| \leq |\lambda_1|$ for any λ_1 that occurs in a homogeneous binomial difference $\lambda_1 - \lambda_2 \in \mathcal{D}$. Consider the following three sets: $\mathcal{D}(\omega_1) = \{\lambda_1 - \lambda_2 \in \mathcal{D} : \omega_1$ occurs as a scattered subword in $\lambda_1\}$, $\mathcal{D}(\omega_2) = \{\lambda_1 - \lambda_2 \in \mathcal{D} : \omega_2$ occurs as a scattered subword in $\lambda_2\}$, and $\mathcal{D}(0) = \{\lambda_1 - \lambda_2 \in \mathcal{D}$: neither ω_1 nor ω_2 occur as scattered subwords in λ_1 and $\lambda_2\}$. Note that $\mathcal{D} = \{\omega_1 - \omega_2\} \cup \mathcal{D}(\omega_1) \cup \mathcal{D}(\omega_2) \cup \mathcal{D}(0)$. Furthermore, these sets are disjoint. If ω_1 were to occur as a scattered subword in λ_1 and ω_2 occurs as a scattered subword of λ_2, then Lemma 20 shows that $\lambda_1 - \lambda_2$ is an algebraic consequence of commutators, $\omega_1 - \omega_2$, and perhaps some other homogeneous binomial difference in $\mathcal{D}$ consisting of words of length less than $|\lambda_1|$; that is, $\lambda_1 - \lambda_2$ is not minimal. Thus, these sets form a partition of $\mathcal{D}$ and so at least one of $\mathcal{D}(\omega_1)$, $\mathcal{D}(\omega_2)$, and $\mathcal{D}(0)$ must be infinite.

Now let $\lambda_1 - \lambda_2 \in \mathcal{D}(\omega_2)$ and suppose $|\lambda_1| > |\omega_1|$. Since λ_1 does not contain ω_1 as a scattered subword, the number of occurrences k_i of some variable x_i in λ_1 must be less than in ω_1, so the number of occurrences k_j of some other variable x_j must be greater than the number of occurrences in ω_1. Suppose $\mathcal{D}(\omega_2)$ is infinite. Then there exists a difference $\lambda_1' - \lambda_2' \in \mathcal{D}(\omega_2)$ with $|\lambda_1'| > |\lambda_1|$, and furthermore, neither λ_1 nor ω_1 can occur as scattered subwords in λ_1'. Thus the number of occurrences $k_{i'}$ of another variable $x_{i'}$ must be less than in ω_1, and so the number of occurrences $k_{j'}$ of another variable $x_{j'}$ must be greater than in ω_1. This indicates that $\mathcal{D}(\omega_2)$ cannot be infinite: for some l, any homogeneous binomial difference $\gamma_1 - \gamma_2 \in \mathcal{D}(\omega_2)$ such that $|\gamma_1| > l$ must have a first word which contains as a scattered subword some word $\bar{\lambda}_1$ which previously occurred in a homogeneous binomial difference $\bar{\lambda}_1 - \bar{\lambda}_2 \in \mathcal{D}(\omega_2)$ and is thus not minimal. The same argument, *mutatis mutandis*, shows that $\mathcal{D}(\omega_1)$ is also finite.

Consider, then, the set $\mathcal{D}(0)$. Let $\omega_1' - \omega_2' \in \mathcal{D}(0)$ be such that $|\omega_1'| \leq |\lambda_1|$ for any $\lambda_1 - \lambda_2 \in \mathcal{D}(0)$. Note that $\omega_1' - \omega_2'$ must consist of words at least as long as ω_1, and furthermore, in both ω_1' and ω_2' some variables x_{k_1} and x_{k_2} must occur less often than in ω_1 and ω_2 respectively. We may partition $\mathcal{D}(0)$ into sets $\mathcal{D}(\omega_1')$, $\mathcal{D}(\omega_2')$, and $\mathcal{D}(0')$ which form a partition of $\mathcal{D}(0)$. As above, these sets form a partition of $\mathcal{D}(0)$, and following the argument above, both $\mathcal{D}(\omega_1')$ and $\mathcal{D}(\omega_2')$ are finite. Consider then $\mathcal{D}(0')$, which must be infinite, and select a difference $\omega_1'' - \omega_2'' \in \mathcal{D}(0')$ such that $|\omega_1''| \leq |\lambda_1|$ for any $\lambda_1 - \lambda_2 \in \mathcal{D}(0')$. Again $\omega_1'' - \omega_2''$ must consist of words at least as long as ω_1', and furthermore, in both ω_1'' and ω_2'' some variables $x_{k_1'}$ and $x_{k_2'}$ must occur less often than in ω_1' and ω_2' respectively. Continuing this partitioning process *ad infinitum* is impossible: for some l, any difference $\gamma_1 - \gamma_2$ such that $|\gamma_1| > l$ must contain the occurrence

of some $\bar{\lambda}_i$, $i \in \{1, 2\}$, which previously occurred in a homogeneous binomial difference in $\bar{\lambda}_1 - \bar{\lambda}_2 \in \mathcal{D}(0)$ as a scattered subword. Thus $\mathcal{D}(0)$ cannot be infinite, and so $\mathcal{M}_A$ is finite and I_A must be finitely generated. $\square$

We have the following corollaries. Corollary 23 gives a description of those homogeneous binomial differences in the commutator ideal. Note that necessity in Corollary 23 was proved in Hinson (2010).

Corollary 22 *Let A be a regular array with pairwise-coprime first column entries. Then $I_A = \mathcal{C}$, where $\mathcal{C}$ denotes the commutator ideal.*

PROOF. Since the entries in the first column of A are pairwise-coprime, $\mathcal{M}_A$ is trivial. $\square$

Corollary 23 *Let A be a regular array with pairwise-coprime first column entries, and suppose $\omega_1, \omega_2 \in \mathcal{B}$ with $|\omega_1| = |\omega_2| = l$. Then $\omega_1 - \omega_2 \in I_A \iff$ supp(ω_1) = supp(ω_2) and $\#(x_i, \omega_1) = \#(x_i, \omega_2)$ for all $x_i \in$ supp(ω_1) = supp(ω_2).*

PROOF. To prove sufficiency, let

$$A = \begin{pmatrix} a_{1,0} & d_1 a_{1,0} & \cdots & d_n a_{1,0} & \cdots \\ a_{2,0} & d_1 a_{2,0} & \cdots & d_n a_{2,0} & \cdots \\ \vdots & \vdots & & \vdots & \\ a_{t,0} & d_1 a_{t,0} & \cdots & d_n a_{t,0} & \cdots \end{pmatrix}.$$

Assume that $\omega_1 - \omega_2 \in I_A$. Then

$$\sigma(\omega_1) = \sigma(\omega_2) \Rightarrow \prod_{i=0}^{l-1} a_{u_i i} = \prod_{i=0}^{l-1} a_{v_i i}.$$

Expressing each weight as a multiple of a first-column entry and canceling the d_i's common to each side of the equation gives

$$\prod_{i=0}^{l-1} a_{u_i 0} = \prod_{i=0}^{l-1} a_{v_i 0}. \tag{5}$$

Since the first column entries of A are pairwise-coprime, equality can only hold in Equation (5) when there exists a bijection between $\{a_{u_i 0}\}_{i=0}^{l-1}$ and $\{a_{v_i 0}\}_{i=0}^{l-1}$.

Each $a_{u_i 0}$ corresponds to an occurrence of the letter x_{u_i} in ω_1; thus, $\mathrm{supp}(\omega_1) = \mathrm{supp}(\omega_2)$, and because the weights are equal, $\#(x_i, \omega_1) = \#(x_i, \omega_2)$ for each $x_i \in \mathrm{supp}(\omega_1) = \mathrm{supp}(\omega_2)$. $\square$

In particular, when a regular array A has pairwise-coprime first column entries, then the algebra R/I_A on which it defines an order is in fact isomorphic to the commutative polynomial algebra $k[x_1, \cdots, x_t]$. The preceding construction can be viewed as an alternative way to specify an admissible length-dominant weight order on the monomials in $k[x_1, \ldots, x_t]$. L. Robbiano has proven that any such order is a lexicographic product of weight orders (Robbiano, 1986).

More generally, in Gilmer (1984), R. Gilmer proved that monoid algebras of commutative monoids are precisely the homomorphic images of polynomial rings by ideals which are generated by pure binomial differences. Of course, the algebra R/I_A is precisely such an algebra when A is regular. Thus, we can view R/I_A as a monoid algebra of a commutative monoid for which an admissible order on the basis of R/I_A can be obtained.

While Theorem 19 shows how to decompose a homogeneous binomial difference over the set of commutators and $\mathcal{M}_A$ and Theorem 21 demonstrates that $\mathcal{M}_A$ is finite, constructing $\mathcal{M}_A$ may present significant computational difficulties. Neglecting minimality, the process of directly identifying a homogeneous binomial difference of disjoint support consisting of words of length l by calculating weights is easily seen to be equivalent to the Subset Product problem, which is known to be NP-complete (Garey & Johnson, 1979). Furthermore, the above proofs do not give a bound on the lengths of the words that may occur in a minimal homogeneous binomial difference of disjoint support, suggesting that even if one is able to devise an algorithm to efficiently identify minimal homogeneous binomial differences of disjoint support, one could not terminate the algorithm and be satisfied that all the minimal homogeneous binomial differences of disjoint support had been enumerated.

3 Weight Ideals Associated to Log-Linear Arrays

Bijectively related to the family of linear arrays is the family of log-linear arrays.

Definition 24 *An array* $A = (a_{ij}) \in \mathcal{M}_{t \times \infty}(S_{>0})$ *is* log-linear *if the array* $\log A = (\log(a_{ij}))$ *is linear.*

Example 25 *The array*

$$B = \begin{pmatrix} e^2 & e^6 & e^{18} & \cdots \\ e^3 & e^9 & e^{27} & \cdots \\ e^4 & e^{12} & e^{36} & \cdots \end{pmatrix}$$

is a log-linear array, because

$$\log B = \begin{pmatrix} 2 & 6 & 18 & \cdots \\ 3 & 9 & 27 & \cdots \\ 4 & 12 & 36 & \cdots \end{pmatrix}$$

is a linear array (with slope $d = 3$).

Note that an array A for which $\log A$ is regular but not linear need not be admissible. Because every log-linear array with slope 1 is in fact a (constant) regular array, we will assume without comment in the remainder that any log-linear array considered has slope $d \neq 1$. We point out also that the base of a log-linear array is immaterial in determining the order given by the array. To see this, let $A \in \mathcal{M}_{t\times\infty}(S_{>0})$ be a given linear array with i, j^{th} entry $d^j a_{i,0}$ and consider the arrays $B = (b_{ij})$ and $C = (c_{ij})$, where $b_{ij} = b^{d^j a_{i,0}}$ and $c_{ij} = c^{d^j a_{i,0}}$ for elements b, $c \in S_{>0}$ with $b \neq c$. Suppose that $\omega_1 = x_{u_0}x_{u_1}\ldots x_{u_{l-1}}$ and $\omega_2 = x_{v_0}x_{v_1}\ldots x_{v_{l-1}}$ are words in X^* such that $\omega_1 \succ_B \omega_2$. Then

$$\sigma_B(\omega_1) > \sigma_B(\omega_2)$$

which means that

$$\prod_{k=0}^{l-1} b_{u_k,k} > \prod_{i=0}^{l-1} b_{v_k,k}.$$

Of course, this is equivalent to the inequality

$$b^{\sum_{k=0}^{l-1} d^k a_{u_k}} > b^{\sum_{k=0}^{l-1} d^k a_{v_k}},$$

and replacing b with c does not change the direction of the inequality. Thus, we will typically assume without comment that the base of a log-linear array is e. As the preceding discussion indicates, the significant distinction between regular arrays and log-linear arrays is that when working with regular arrays, the weight associated to a word is calculated by multiplying the weights given to each variable in their respective positions, while when working with log-linear arrays, the weight associated to a word is calculated by adding the weights given to each variable in their respective positions. In particular, when working with a log-linear array A, the equation $\sum_{k=0}^{l-1} d^k a_{u_k} = \sum_{k=0}^{l-1} d^k a_{v_k}$ must be satisfied for a homogeneous binomial difference $\omega_1 - \omega_2 = x_{u_0}x_{u_1}\ldots x_{u_{l-1}} - x_{v_0}x_{v_1}\ldots x_{v_{l-1}}$ to be a member of I_A.

The structure of log-linear arrays is much less uniform than that of regular arrays. Theorem 26 is the main result of this section and is proved via the examples that follow. We will also demonstrate that log-linear arrays can be constructed to give orders on R which are equivalent to the familiar left and right length-lexicographic orders.

Theorem 26 *There exist log-linear arrays whose associated weight ideals are trivial, log-linear arrays whose associated weight ideals admit a finite generating set, and log-linear arrays whose associated weight ideals do not admit a finite generating set.*

It would be of interest to find necessary and sufficient conditions on a log-linear array A such that I_A is trivial, is nontrivial but admits a finite generating set, or is nontrivial and does not admit a finite generating set.

The following two lemmas describe distinct arrays that define orders on R which are equivalent to left and right length-lexicographic order respectively. The hypotheses on the array A is sufficient to insure in each case that I_A is trivial.

Lemma 27 *Suppose the variables $x_1, \ldots, x_t$ are ordered. Let $A \in \mathcal{M}_{t\times\infty}(S_{>0})$ be log-linear with first column $A_{(0)} = [e^{a_{1,0}}, \ldots, e^{a_{t,0}}]^T$ for which the values of the first-column entries of A reflect the order given to $x_1, \ldots, x_t$; that is, $x_{i_1} < x_{i_2}$ if and only if $a_{i_1,0} < a_{i_2,0}$ also. Let α and β denote the minimum and maximum nonzero first column differences of $\log A$ respectively; that is,*

$$\alpha = \min\{|a_{i,0} - a_{j,0}| : 1 \le i, j \le t, i \ne j\},$$

and

$$\beta = \max\{|a_{i,0} - a_{j,0}| : 1 \le i, j \le t, i \ne j\}.$$

Let d be the slope of $\log A$. If $d < 1$ and $\alpha > d\beta/(1-d)$, then I_A is trivial and the order given by A is the left length-lexicographic order.

PROOF. Assume the hypotheses, and assume that $\omega_1 = x_{u_0}x_{u_1}\ldots x_{u_{l-1}}$ and $\omega_2 = x_{v_0}x_{v_1}\ldots x_{v_{l-1}}$ are two words of equal length in R such that $\omega_1 - \omega_2 \in I_A$. This implies that

$$e^{a_{u_0,0}+da_{u_1,0}+\cdots+d^{l-1}a_{u_{l-1},0}} = e^{a_{v_0,0}+da_{v_1,0}+\cdots+d^{l-1}a_{v_{l-1},0}}, \tag{6}$$

and thus

$$a_{u_0,0} + da_{u_1,0} + \cdots + d^{l-1}a_{u_{l-1},0} = a_{v_0,0} + da_{v_1,0} + \cdots + d^{l-1}a_{v_{l-1},0}.$$

This in turn implies that

$$a_{u_0,0} - a_{v_0,0} = \sum_{i=1}^{l-1} d^i \left(a_{u_i,0} - a_{v_i,0}\right). \tag{7}$$

Suppose now that the first letters of ω_1 and ω_2 differ. The largest in absolute value that the right-hand side of Equation 7 can be is when each difference $a_{u_i,0} - a_{v_i,0} = \beta$, so the right-hand side has an upper bound at $\beta(d-d^l)/(1-d)$. The smallest the left-hand side of Equation 7 can be in absolute value is when $a_{u_0,0} - a_{v_0,0} = \alpha$, and by hypothesis, $\alpha > d\beta/(1-d) > (d-d^l)\beta/(1-d)$ for all l. This contradicts the assumption that $\omega_1 - \omega_2 \in I_A$, so in fact $I_A = \{0\}$. Furthermore, the difference $a_{u_0,0} - a_{v_0,0}$ is greater in absolute value than any possible subsequent sum and hence determines the order between ω_1 and ω_2.

Now, note that if the first k letters of ω_1 and ω_2 are the same, then those first k letters contribute the same expression to either side of Equation 6, and thus play no role in determining the order between ω_1 and ω_2. Thus, when the first k letters are the same, we may determine the order between ω_1 and ω_2 by simply applying the above argument to the truncated words $x_{u_k} x_{u_{k+1}} \cdots x_{u_{l-1}}$ and $x_{v_k} x_{v_{k+1}} \cdots x_{v_{l-1}}$ to note that the order on ω_1 and ω_2 is determined solely by the order between $a_{u_k,0}$ and $a_{v_k,0}$.

Because the order on the first column entries of $\log A$ is equivalent to the order on the variables $x_1, \ldots, x_t$ the order given by A is thus the left length-lexicographic order. $\square$

Lemma 28 *Suppose the variables $x_1, \ldots, x_t$ are ordered. Let $A \in \mathcal{M}_{t\times\infty}(S_{>0})$ be log-linear with first column $A_{(0)} = [e^{a_{1,0}}, \ldots, e^{a_{t,0}}]^T$ for which the values of the first column entries of A reflect the order given to $x_1, \ldots, x_t$; that is, $x_{i_1} < x_{i_2}$ if and only if $a_{i_1,0} < a_{i_2,0}$ also. Let α and β denote the minimum and maximum nonzero first column differences of $\log A$ respectively; that is,*

$$\alpha = \min\{|a_{i,0} - a_{j,0}| : 1 \le i, j \le t, i \ne j\},$$

and

$$\beta = \max\{|a_{i,0} - a_{j,0}| : 1 \le i, j \le t, i \ne j\}.$$

If $d > 1$ and $\alpha > \beta/(d-1)$, then I_A is trivial and the order given by A is the right length-lexicographic order.

PROOF. The same argument as in Lemma 27 applies, *mutatis mutandis*.

Orders constructed via admissible arrays with trivial weight ideals are simply admissible length-dominant orders on R. A set of invariants that fully characterize the admissible orders that can be defined on a noncommutative

k-algebra such as R has not yet been described, though results in this direction have been obtained (Scott, 1994; Perlo-Freeman & Pröhle, 1997). It is possible that array-based admissible orders may be of use in defining such a set of invariants.

We turn our attention next to an example of a log-linear array A for which I_A is nontrivial and admits a finite generating set.

Example 29 *Let A be the log-linear array such that*

$$\log A = \begin{pmatrix} 2 & 4 & 8 & \dots \\ 3 & 6 & 12 & \dots \\ 4 & 8 & 16 & \dots \\ 6 & 12 & 24 & \dots \end{pmatrix}.$$

The weight ideal I_A associated to A is nontrivial and is finitely generated.

Clearly the weight ideal associated to A is nontrivial; for example, $x_1x_2 - x_3x_1 \in I_A$. Interestingly, any homogeneous binomial difference of length $l > 2$ in I_A can be reduced in at most two steps to an algebraic consequence of homogeneous binomial differences consisting of words whose maximum length is $l - 1$. It follows inductively that any homogeneous binomial difference of length l can be reduced to an algebraic consequence of homogeneous binomial differences of length 2; that is, for this particular A, the homogeneous binomial differences of length 2 in fact generate I_A. The proof of this proposition is straightforward but relies on a lengthy case-by-case analysis, which is included as an appendix.

The next example demonstrates that there exists log-linear arrays with a nontrivial associated weight ideal which admits no finite generating set.

Example 30 *Let A be the log-linear array given by*

$$\log A = \begin{pmatrix} 2 & 4 & 8 & \dots \\ 4 & 8 & 16 & \dots \\ 7 & 14 & 28 & \dots \end{pmatrix}.$$

The weight ideal I_A associated to A is nontrivial and does not admit a finite generating set.

In the proof we will use the term *factor* to indicate a subword in which the letters occur consecutively, in order to alleviate any potential confusion with scattered subwords, which some authors refer to simply as subwords.

PROOF. Consider the homogeneous binomial difference

$$\omega_1 - \omega_2 := x_2 x_3^n x_2 - x_1 (x_2 x_3)^{(n-2)/2} x_1 x_2 x_3,$$

where $n \geq 4$ is an even integer and $l = n + 2$ is the length of ω_1 and ω_2. We will show first that for any such n, the difference given above is a member of I_A, and then we will demonstrate that $\omega_1 - \omega_2$ is not an algebraic consequence of the shorter length differences in I_A and so must belong to any generating set for I_A. Since this holds for all $n \geq 4$, this will prove that I_A does not admit a finite generating set.

To demonstrate that $x_2 x_3^n x_2 - x_1 (x_2 x_3)^{(n-2)/2} x_1 x_2 x_3 \in I_A$ for any $n \geq 4$, let us calculate the difference of the weights associated to $\omega_1 - \omega_2$ respectively. Fix an even integer $n \geq 4$. The difference in weights associated to any homogeneous binomial difference by A can be expressed as a polynomial:

$$\Delta := a_0 + d a_1 + d^2 a_2 + \cdots + d^{l-1} a_{l-1},$$

where a_k is the difference of the first-column entries associated to the letter in position k in ω_1 and ω_2 respectively. For the given difference,

$$\Delta_{\omega_1 - \omega_2} = 2 + 2 \cdot 3 + 2^2 \cdot 0 + 2^3 \cdot 3 + 2^4 \cdot 0 + \cdots + 2^{l-4} \cdot 0 + 2^{l-3} \cdot 4 + 2^{l-2} \cdot 3 + 2^{l-1} \cdot (-3).$$

To show that $\Delta_{\omega_1 - \omega_2} = 0$ regardless of the value of n, it is easiest to work in binary. We have

$$\begin{aligned} &10 + 10 \cdot 11 + 1000 \cdot 11 + 100000 \cdot 11 + \cdots \\ &\qquad + 1\underbrace{0\ldots0}_{l-5} \cdot 11 + 1\underbrace{0\ldots0}_{l-3} \cdot 101 + 1\underbrace{0\ldots0}_{l-2} \cdot 11 - 1\underbrace{0\ldots0}_{l-1} \cdot 11. \end{aligned}$$

Multiplying simplifies this to

$$10 + 110 + 11000 + 1100000 + \cdots + 11\underbrace{0\ldots0}_{l-3} + 101\underbrace{0\ldots0}_{l-3} + 11\underbrace{0\ldots0}_{l-2} - 11\underbrace{0\ldots0}_{l-1}.$$

This expression is equal to 0:

$$1\underbrace{0\ldots0}_{l-3} + 101\underbrace{0\ldots0}_{l-3} + 11\underbrace{0\ldots0}_{l-2} - 11\underbrace{0\ldots0}_{l-1}$$

and so

$$\begin{gathered} 11\underbrace{0\ldots0}_{l-2} + 11\underbrace{0\ldots0}_{l-2} - 11\underbrace{0\ldots0}_{l-1} = \\ 11\underbrace{0\ldots0}_{l-1} - 11\underbrace{0\ldots0}_{l-1} = 0. \end{gathered}$$

This demonstrates that $x_2 x_3^n x_2 - x_1 (x_2 x_3)^{(n-2)/2} x_1 x_2 x_3 \in I_A$. To show that $x_2 x_3^n x_2 - x_1 (x_2 x_3)^{(n-2)/2} x_1 x_2 x_3$ must be contained in any generating set for

I_A, we will show that the word $x_2x_3^nx_2$ contains no factor that occurs as a word in a homogeneous binomial difference of shorter length in I_A. In particular, this implies that $x_2x_3^nx_2 - x_1(x_2x_3)^{(n-2)/2}x_1x_2x_3$ cannot be written as an algebraic consequence of strictly shorter length homogeneous differences in I_A.

Consider the possible factors of the word $x_2x_3^nx_2$. For each k, $4 \le k \le n$, we have factors $x_2x_3^k$, factors $x_3^kx_2$, and x_3^k. No factor that occurs in a homogeneous binomial difference in I_A can begin with x_3, because of the parity of the weight that results, so we can rule out as possibilities any factors of the form x_3^k and $x_3^kx_2$. The weight given the factor $x_2x_3^k$ will be greater than the weight assigned any other word of equal length except x_3^k, and it will not equal this weight. Thus, $x_2x_3^nx_2$ contains no factors that occur as a word in a homogeneous binomial difference in I_A. Because this holds for each $n \ge 4$, the difference $x_2x_3^nx_2 - x_1(x_2x_3)^{(n-2)/2}x_1x_2x_3$ must be included in any generating set for I_A, and thus any generating set for I_A is infinite. $\square$

A Proof that the Array in Example 29 is Finitely Generated

PROOF. Let us first list the the homogeneous binomial differences of length two that occur in I_A. They are $x_1x_2 - x_3x_1$, $x_1x_3 - x_3x_2$, $x_1x_3 - x_4x_1$, $x_3x_2 - x_4x_1$, $x_3x_3 - x_4x_2$, and $x_1x_4 - x_4x_3$. Given a homogeneous binomial difference consisting of words of length l in I_A, we will show that it either decomposes over the homogeneous binomial differences in I_A of length two or fails to belong to I_A. Proof of the latter claim requires us to consider the existence of solutions to the polynomial

$$a_0 + da_1 + \cdots + d^{l-1}a_{l-1} = 0$$

which corresponds to a given homogenous binomial difference in I_A. Any solution to this equation is an element of the set $\coprod_{i=0}^{l-1} A_d = \{(a_0, \ldots, a_{l-1}) | a_i = a_{j,0} - a_{k,0}, 1 \le j, k \le 4\}$.

To simplify exposition, we use the notation of rewriting relations on words in the free monoid $X^* := \langle x_1, \ldots, x_t \rangle$. We define the following rewriting relation: for $\omega_1, \omega_2 \in X^*$, we write $\omega_1 \overset{*}{\longleftrightarrow} \omega_2$ to denote that $\omega_1 - \omega_2 \in I_A$. A particular chain of rewritings $\omega_1 \overset{*}{\longleftrightarrow} \lambda_1 \overset{*}{\longleftrightarrow} \cdots \overset{*}{\longleftrightarrow} \lambda_n \overset{*}{\longleftrightarrow} \omega_2$ corresponds to a unique decomposition of $\omega_1 - \omega_2$ in R (Madlener & Reinert, 1997), though this decomposition need not be over homogeneous binomial differences consisting of words of lesser length. However, a chain of rewritings $\omega_1 \overset{*}{\longleftrightarrow} \lambda$ where at least the first letter of λ is the same as the first letter of ω_2 does correspond to a unique decomposition of $\omega_1 - \omega_2$ over the set of homogeneous binomial differences in I_A whose words are of lesser length. Rather than calculate the decomposition precisely, we will rewriting to indicate that a decomposition is possible.

Organizing by weight, the length two homogeneous binomial differences in I_A give rise to the following rewriting relations:

$$x_1x_2 \overset{*}{\longleftrightarrow} x_3x_1, \qquad x_1x_3 \overset{*}{\longleftrightarrow} x_3x_2 \overset{*}{\longleftrightarrow} x_4x_1,$$

$$x_3x_3 \overset{*}{\longleftrightarrow} x_4x_2, \qquad x_1x_4 \overset{*}{\longleftrightarrow} x_4x_3.$$

Suppose that $\omega_1 - \omega_2 \in I_A$, with $|\omega_1| = |\omega_2| = l$. Let $\omega_1 = x_{u_0} \dots x_{u_{l-1}}$ and $\omega_2 = x_{v_0} \dots x_{v_{l-1}}$. By parity, if either ω_1 or ω_2 start with x_2, then so must the other, in which case $\omega_1 - \omega_2$ is immediately reducible to a homogeneous binomial difference of length $l-1$. Thus we may assume without loss of generality that neither ω_1 nor ω_2 start with x_2.

Assume next that ω_1 begins with x_1. We need to consider the cases when ω_2 starts with x_3 or x_4, as well as the case when ω_1 begins with x_3 and ω_2 begins with x_4. All other cases will then be captured by symmetry.

If $\omega_1 - \omega_2$ is not immediately reducible, then ω_2 must begin with either x_3 or x_4.

Case 1 : ω_2 begins with x_3.

Subcase 1.1 : ω_2 begins with x_3x_1.

$x_3x_1 \overset{*}{\longleftrightarrow} x_1x_2$, so $\omega_1 - \omega_2$ is reducible after this single rewrite.

Subcase 1.2 : ω_2 begins with x_3x_2.

$x_3x_2 \overset{*}{\longleftrightarrow} x_1x_3$, so $\omega_1 - \omega_2$ is reducible after this single rewrite.

Subcase 1.3 : ω_2 begins with x_3x_3.

In this case, reduction with a single rewrite of ω_2 is not always possible. If the second letter of ω_1 is x_2 or x_3, then we can rewrite ω_1 to reduce $\omega_1 - \omega_2$. Suppose then that the second letter of ω_1 is x_4. Rewrite as follows:

$$x_3x_3 \overset{*}{\longleftrightarrow} x_4x_2 \text{ and } x_1x_4 \overset{*}{\longleftrightarrow} x_4x_3.$$

Now first letters agree and $\omega_1 - \omega_2$ is reducible. Finally, suppose that the second letter of ω_1 is x_1. Then $\omega_1 - \omega_2$ is of the form

$$x_1x_1x_{u_2} \dots x_{u_{l-1}} - x_3x_3x_{v_2} \dots x_{v_{l-1}}.$$

Since $\omega_1 - \omega_2 \in I_A$, this gives rise to the equation $a_0 + 2a_1 + \cdots + 2^{l-1}a_{l-1} = 0$, where a_0 is a difference of first-column entries of I_A determined by the letters

of ω_1 and ω_2. In particular, $a_0 = -2$ and $a_1 = -2$, so

$$-2 - 4 + 2^2 a_2 + \cdots + 2^{l-1} a_{l-1} = 0,$$

or equivalently,

$$2a_2 + \cdots + 2^{l-2} a_{l-1} = 3, \tag{A.1}$$

but (A.1) does not have a solution over $\coprod_{i=0}^{l-1} A_d$.

Subcase 1.4 : ω_2 begins with $x_3 x_4$.

Again, if the second letter of ω_1 is x_2 or x_3, then $\omega_1 - \omega_2$ is immediately reducible. Suppose the second letter of ω_1 is x_1. Then $\omega_1 - \omega_2$ is of the form

$$x_1 x_1 x_{u_2} \ldots x_{u_{l-1}} - x_3 x_4 x_{v_2} \ldots x_{v_{l-1}}.$$

Since $\omega_1 - \omega_2 \in I_A$, this gives rise to the equation $a_0 + 2a_1 + \cdots + 2^{l-1} a_{l-1} = 0$, where each a_i is a difference of first column entries of A determined by the letters of ω_1 and ω_2. In particular, $a_0 = -2$ and $a_1 = -4$, so

$$-2 - 8 + 2^2 a_2 + \cdots + 2^{l-1} a_{l-1} = 0,$$

or equivalently,

$$2a_2 + \cdots + 2^{l-2} a_{l-1} = 5, \tag{A.2}$$

but (A.2) does not have a solution over $\coprod_{i=0}^{l-1} A_d$.

Finally, if the second letter of ω_1 is x_4, then $\omega_1 - \omega_2$ is of the form

$$x_1 x_4 x_{u_2} \ldots x_{u_{l-1}} - x_3 x_4 x_{v_2} \ldots x_{v_{l-1}}.$$

Since $\omega_1 - \omega_2 \in I_A$, this gives rise to the equation $a_0 + 2a_1 + \cdots + 2^{l-1} a_{l-1} = 0$, where each a_i is a difference of first column entries of A determined by the letters of ω_1 and ω_2. In particular, $a_0 = -2$ and $a_1 = 0$, so

$$-2 - 0 + 2^2 a_2 + \cdots + 2^{l-1} a_{l-1} = 0,$$

or equivalently,

$$2a_2 + \cdots + 2^{l-2} a_{l-1} = 1, \tag{A.3}$$

but (A.3) does not have a solution over $\coprod_{i=0}^{l-1} A_d$.

Case 2: ω_1 begins with x_1 and ω_2 begins with x_4.

Subcase 2.1: ω_1 begins with x_1 and ω_2 begins with $x_4 x_1$.

$x_4 x_1 \overset{*}{\longleftrightarrow} x_1 x_3$, so $\omega_1 - \omega_2$ is reducible after this single rewrite.

Subcase 2.2: ω_1 begins with x_1 and ω_2 begins with $x_4 x_2$.

If ω_1 begins with x_1x_3, then we can rewrite $x_1x_3 \overset{*}{\longleftrightarrow} x_4x_1$ and immediately reduce $\omega_1 - \omega_2$. Similarly, if ω_1 begins with x_1x_4, we can immediately rewrite $x_1x_4 \overset{*}{\longleftrightarrow} x_4x_3$ to reduce $\omega_1 - \omega_2$. If ω_1 begins with x_1x_2, rewrite $x_1x_2 \overset{*}{\longleftrightarrow} x_3x_1$ and $x_4x_2 \overset{*}{\longleftrightarrow} x_3x_3$ to reduce $\omega_1 - \omega_2$. The only remaining possibility is that ω_1 begins with x_1x_1 and ω_2 begins with x_4x_2, but the corresponding polynomial shows that no such difference that begins with these letters can belong to I_A:

$$-4 + 2 \cdot (-1) + 2^2 a_2 + \cdots + 2^{l-1} a_{l-1} = 0$$

reduces to

$$-3 + 2a_2 + \cdots + 2^{l-2} a_{l-1} = 0,$$

and this equation has no solution over $\coprod_{i=0}^{l-1} A_d$.

Subcase 2.3: ω_1 begins with x_1 and ω_2 begins with x_4x_3.

$x_4x_3 \overset{*}{\longleftrightarrow} x_1x_4$, so $\omega_1 - \omega_2$ is reducible after this single rewrite.

Subcase 2.4: ω_1 begins with x_1 and ω_2 begins with x_4x_4.

Subsubcase 2.4.1: ω_1 begins with x_1x_1.

Consider the polynomial equation $a_0 + 2a_1 + \cdots + 2^{l-1}a_{l-1} = 0$ corresponding to this difference. The choice of first letters for ω_1 and ω_2 determine $a_0 = -4$ and $a_1 = -4$. Factoring out 4 from the resulting equation gives

$$-3 + a_2 + 2a_3 + \cdots + 2^{l-3} a_{l-1} = 0.$$

Each term after the second in the expression on the left–hand side of the above equation is congruent to 0 mod 2, thus this equation has a solution only if $a_2 = \pm 3$ or ± 1 (each $a_i \in \{\pm 1, \pm 2, \pm 3 \pm 4\}$. There are thus four possible subcases to consider. If $a_2 = -3$, then the first three letters of ω_1 are $x_1x_1x_2$, and $x_2x_1x_2 \overset{*}{\longleftrightarrow} x_1x_3x_1 \overset{*}{\longleftrightarrow} x_4x_1x_1$, so $\omega_1 - \omega_2$ is reducible. If $a_2 = 3$, the first three letters of ω_1 are $x_1x_1x_4$, and $x_1x_1x_4 \overset{*}{\longleftrightarrow} x_1x_4x_3 \overset{*}{\longleftrightarrow} x_4x_4x_3$, so $\omega_1 - \omega_2$ is reducible. If $a_2 = -1$ then either ω_1 begins with $x_1x_1x_1$ or $x_1x_1x_2$. In the first case, ω_2 must therefore begin with $x_4x_4x_2$, and $x_4x_4x_2 \overset{*}{\longleftrightarrow} x_4x_3x_3 \overset{*}{\longleftrightarrow} x_1x_4x_3$, so the difference is reducible, and in the second case, $x_1x_1x_2 \overset{*}{\longleftrightarrow} x_1x_3x_1 \overset{*}{\longleftrightarrow} x_4x_1x_1$, so the difference is reducible. If $a_2 = 1$, then either ω_1 begins with $x_1x_1x_2$ or ω_1 begins with $x_1x_1x_3$. The first case has already been addressed, and the second case gives rise to a reducible instance of $\omega_1 - \omega_2$, for $x_1x_1x_3 \overset{*}{\longleftrightarrow} x_1x_3x_2 \overset{*}{\longleftrightarrow} x_4x_1x_2$.

Subsubcase 2.4.2: ω_1 begins with x_1x_2.

The polynomial equation corresponding to this difference is

$$-4 + 2(-3) + \cdots + 2^{l-1} a_{l-1} = 0,$$

and this equation has no solution over $\coprod_{i=0}^{l-1} A_d$.

Subsubcase 2.4.3: ω_1 begins with x_1x_3.

$x_1x_3 \overset{*}{\longleftrightarrow} x_4x_3$, so $\omega_1 - \omega_2$ is reducible after this single rewrite.

Case 3: ω_1 begins with x_3 and ω_2 begins with x_4.

Subcase 3.1: ω_1 begins with x_3 and ω_2 begins with x_4x_1.

$x_4x_1 \overset{*}{\longleftrightarrow} x_3x_2$, so $\omega_1 - \omega_2$ is reducible after this single rewrite.

Subcase 3.2: ω_1 begins with x_3 and ω_2 begins with x_4x_2.

$x_4x_2 \overset{*}{\longleftrightarrow} x_3x_3$, so $\omega_1 - \omega_2$ is reducible after this single rewrite.

Subcase 3.3: ω_1 begins with x_3 and ω_2 begins with x_4x_3.

There are a number of cases to consider. If ω_1 begins with x_3x_1, since $x_3x_1 \overset{*}{\longleftrightarrow} x_1x_2$ and $x_4x_3 \overset{*}{\longleftrightarrow} x_1x_4$, the difference is reducible. If ω_1 begins with x_3x_2, we can rewrite $x_3x_2 \overset{*}{\longleftrightarrow} x_1x_3$ and $x_4x_3 \overset{*}{\longleftrightarrow} x_1x_4$ to reduce $\omega_1 - \omega_2$. If ω_1 begins with x_3x_3, we may rewrite $x_3x_3 \overset{*}{\longleftrightarrow} x_4x_2$ to immediately reduce $\omega_1 - \omega_2$. It remains to consider the case when ω_1 begins with x_3x_4 and ω_2 begins with x_4x_3. The corresponding polynomial shows that a homogeneous binomial difference that starts with these letters cannot occur in I_A:

$$-2 + 2 \cdot (2) + 2^2 a_2 + \cdots + 2^{l-1} a_{l-1} = 0$$

implies

$$1 + 2a_2 + \cdots + 2^{l-2} a_{l-1} = 0,$$

and this equation does not have a solution over $\coprod_{i=0}^{l-1} A_d$.

Subcase 3.4: ω_1 begins with x_3 and ω_2 begins with x_4x_4.

The corresponding polynomial equation for a difference with these starting letters is

$$-2 + 2a_1 + \cdots + 2^{l-1} a_{l-1} = 0,$$

and we can factor out the common 2 to obtain

$$-1 + a_1 + \cdots + 2^{l-1} a_{l-1} = 0.$$

In order for this equation to have a solution, $a_1 \in \{\pm 1, \pm 3\}$, but because the second letter of ω_2 is x_4, $a_1 \neq \pm 1$ and $a_1 \neq 3$, so $a_1 = -3$. Thus ω_1 begins with x_3x_2, and because $x_3x_2 \overset{*}{\longleftrightarrow} x_4x_1$, $\omega_1 - \omega_2$ is reducible. $\square$

References

Adian, S.I., Durnev, V.G., 2000. Decision Problems for Groups and Semigroups. Russian Math. Surveys, vol. 55 no. 2, 207-296.

Gilmer, R., 1984. Commutative Semigroup Rings, Chicago Lect. in Math., The University of Chicago Press, Chicago.

Green, E.L., 1990. GRB. Available from: http://www.math.vt.edu/people/green/grb/.

Green, E.L., 1996. Noncommutative Gröbner Bases: A Computational and Theoretical Tool. Lecture Notes, Holiday Mathematics Symposium, New Mexico State University.

Green, E.L., 2000. Multiplicative Bases, Gröbner Bases, and right Gröbner Bases. Journal of Symbolic Computation, Vol. 29, 601 - 623.

Garey, M., Johnson, D., 1979. Computers and Intractability: A Guide to the Theory of NP-Completeness. Addison-Wesley.

Hinson, E.K., 2010. Admissibility of Weight-Defined Term Orders on Quotients of Free Associative Algebras. Preprint.

Madlener, K., Reinert, B., 1998. Relating Rewriting Techniques on Monoids and Rings: Congruences on Monoids and Ideals in Monoid Rings. Journal of Theoretical Computer Science, Vol. 208, 3 - 31.

Nordbeck, P., 2001. Canonical Bases for Algebraic Computations, Ph. D Thesis, Lund Institute of Technology.

Mora, T., 1994. An Introduction to Commutative and Non-Commutative Gröbner Bases. Theoretical Computer Science, vol. 134, 131 - 173.

Perlo-Freeman, S., Pröhle, P., 1997. Scott's Conjecture is True, Position Sensitive Weights. Proceedings of the 8th International Conference on Rewriting Techniques and Applications Lecture Notes In Computer Science, Vol. 1232, 217 - 227.

Robbiano, L., 1986. On the Theory of Graded Structures. Journal of Symbolic Computation, Vol. 2, 139 - 170.

Scott, E.A., 1994. Weights for Total Division Orderings on Strings. Theoretical Computer Science, Vol. 135, pp.345 - 359.